6.1 – Inner Products

Definition: An **inner product** on a real vector space V is a function that associates a real number $\langle \mathbf{u},\ \mathbf{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k.

1. $\langle \mathbf{u},\ \mathbf{v} \rangle = \langle \mathbf{v},\ \mathbf{u} \rangle$ (symmetry axiom)
2. $\langle \mathbf{u} + \mathbf{v},\ \mathbf{w} \rangle = \langle \mathbf{u},\ \mathbf{w} \rangle + \langle \mathbf{v},\ \mathbf{w} \rangle$ (additivity axiom)
3. $\langle k\mathbf{u},\ \mathbf{v} \rangle = k \langle \mathbf{u},\ \mathbf{v} \rangle$ (homogeneity axiom)
4. $\langle \mathbf{v},\ \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v},\ \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ (positivity axiom)

Definition: A real vector space with an inner product is called a **real inner product space**.

Examples of Inner products

- The dot product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$ is the **Euclidean inner product** or **standard inner product** on R^n. [R^n with the Euclidean inner product is called **Euclidean n-space**.]
- If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n and $w_1, w_2, \ldots, w_n$ are positive real numbers, then the formula $\langle \mathbf{u}, \mathbf{v} \rangle = w_1u_1v_1 + w_2u_2v_2 + \ldots + w_nu_nv_n$ is called the **weighted Euclidean inner product with weights** $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}$.
- On M_{nn}, the set of $n \times n$ matrices: If $\mathbf{u} = U$ and $\mathbf{v} = V$ are matrices in the vector space M_{nn}, then the formula $\langle \mathbf{u}, \mathbf{v} \rangle = \mathrm{tr}(U^TV)$ is the **standard inner product** on M_{nn}.
- If $\mathbf{p} = a_0 + a_1x + \ldots + a_nx^n$ and $\mathbf{q} = b_0 + b_1x + \ldots + b_nx^n$ are polynomials in P_n, then the **standard inner product** on P_n is $\langle \mathbf{p}, \mathbf{q} \rangle = a_0b_0 + a_1b_1 + \ldots + a_nb_n$. (Note the similarity in form to the dot product.)
- If $\mathbf{p} = a_0 + a_1x + \ldots + a_nx^n$ and $\mathbf{q} = b_0 + b_1x + \ldots + b_nx^n$ are polynomials in P_n and if $x_0, x_1, \ldots, x_n$ are distinct real numbers, then the formula $\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)\,q(x_0) + p(x_1)\,q(x_1) + \ldots + p(x_n)\,q(x_n)$ is the **evaluation inner product** at $x_0, x_1, \ldots, x_n$.
- If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two functions in $C[a, b]$, then $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)dx$ defines an **inner product** on $C[a, b]$.
- If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n expressed in column form, A is an invertible $n \times n$ matrix, and $\mathbf{u} \cdot \mathbf{v}$ is the Euclidean inner product on R^n, then the formula $\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathrm{v}$ is the **inner product on $\mathbf{R^n}$ generated by A**. This is an example of a **matrix inner product**.

#10 Compute the standard inner product on M_{22} of the given matrices.

$U = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}, V = \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}$

#11 Find the standard inner product on P_2 of the given polynomials.

$\mathbf{p} = -2 + x + 3x^2, \mathbf{q} = 4 - 7x^2$

#16 A sequence of sample points is given. Use the evaluation inner product on P_3 at those sample points to find $\langle \mathbf{p}, \mathbf{q} \rangle$ for the polynomials $\mathbf{p} = x + x^3$ and $\mathbf{q} = 1 + x^2$.
$x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$

#8 Use the inner product on R^2 generated by the matrix A to find $\langle \mathbf{u}, \mathbf{v} \rangle$ for the vectors $\mathbf{u} = (0, -3)$ and $\mathbf{v} = (6, 2)$.

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

Example: A weighted Euclidean inner product

<u>Definition:</u> If V is a real inner product space, then the **norm** or **length** of a vector $\mathbf{v}$ in V is denoted by $\|\mathbf{v}\|$ and is defined by $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v},\ \mathbf{v} \rangle}$ and the **distance** between two vectors is denoted by $d(\mathbf{u},\ \mathbf{v})$ and is defined by $d\,(\mathbf{u},\ \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$. A vector of norm 1 is called a **unit vector**.

#1 Let R^2 have the weighted Euclidean inner product $\langle \mathbf{u},\ \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$ and let $\mathbf{u} = (1,\ 1)$, $\mathbf{v} = (3,\ 2)$, $\mathbf{w} = (0,\ -1)$, and $k = 3$. Compute the stated quantities.
a. $\langle \mathbf{u},\ \mathbf{v} \rangle$ b. $\langle k\mathbf{u},\ \mathbf{v} \rangle$ c. $\langle \mathbf{u} + \mathbf{v},\ \mathbf{w} \rangle$ d. $\|\mathbf{v}\|$ e. $d(\mathbf{u},\ \mathbf{v})$ f. $\|\mathbf{u} - k\mathbf{v}\|$

Theorem 6.1.1 Norms and Distances in Inner Product Spaces
If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a real inner product space V, and if k is a scalar, then:
a) $\|\mathbf{v}\| \geq 0$ with equality if and only if $\mathbf{v} = \mathbf{0}$ (analogous to Theorem 3.2.1 (a) and (b)).
b) $\|k\mathbf{v}\| = |k|\,\|\mathbf{v}\|$ (analogous to Theorem 3.2.1 (c)).
c) $d(\mathbf{u},\ \mathbf{v}) = d(\mathbf{v},\ \mathbf{u})$ (analogous to Theorem 3.2.2 (d)).
d) $d(\mathbf{u},\ \mathbf{v}) \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{v}$ (ibid).

#**37** Let the vector space P_2 have the inner product $\langle \mathbf{p},\ \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx$. Find the following for $\mathbf{p} = 1$ and $\mathbf{q} = x^2$.

a. $\langle \mathbf{p},\ \mathbf{q} \rangle$ b. $d(\mathbf{p},\ \mathbf{q})$ c. $\|\mathbf{p}\|$ d. $\|\mathbf{q}\|$

Definition: If V is an inner product space, then the set of points in V that satisfy $\|\mathbf{u}\| = 1$ is called the **unit sphere** in V (or the **unit circle** in the case where $V = R^2$).

Theorem 6.1.2 Algebraic Properties of Inner Products (generalization of Theorem 3.2.3)

If **u, v** and **w** are vectors in a real inner product space V, and if k is a scalar, then:

a) $\langle \mathbf{0},\ \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{0}\rangle = 0$

b) $\langle \mathbf{u},\ \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{v}\rangle + \langle \mathbf{u},\ \mathbf{w}\rangle$

c) $\langle \mathbf{u},\ \mathbf{v}-\mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{v}\rangle - \langle \mathbf{u},\ \mathbf{w}\rangle$

d) $\langle \mathbf{u}-\mathbf{v},\ \mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{w}\rangle - \langle \mathbf{v},\ \mathbf{w}\rangle$

e) $k\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u},\ k\mathbf{v}\rangle$